

ON RESOLUTION OF COMPACTIFICATIONS OF UNRAMIFIED PLANAR SELF-MAPS

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ABSTRACT. The goal of this paper is to approach the two-dimensional Jacobian Conjecture using ideas of birational algebraic geometry. We study the resolution of rational self-map of the projective plane that comes from a hypothetical counterexample to the two-dimensional Jacobian Conjecture and establish several strong restrictions on its structure. In particular, we get a very detailed description of its Stein factorization. We also establish some combinatorial results on determinants of the Gram matrix of weighted trees and forests and apply them to study exceptional divisorial valuations of the field of rational function in two variables with centers outside of the affine plane. We use some techniques of modern birational geometry, in particular adjunction inequalities, to further restrict the structure of possible counterexamples. Ultimately, we hope that this paper will pave the way for settling the two-dimensional Jacobian Conjecture using the techniques of modern birational geometry.

1. INTRODUCTION

The goal of this paper is to develop a new geometric approach to the two-dimensional Jacobian Conjecture. Some of the results may also be used to understand better the structure of the polynomial automorphisms of the plane. All varieties in this paper are over complex numbers.

This paper is written primarily for the algebraic geometers, by an algebraic geometer. So we take for granted the properties of canonical divisors on normal surfaces, while working out carefully some elementary combinatorial results.

Suppose $f(x, y)$ and $g(x, y)$ are two polynomials with complex coefficients. The classical Jacobian Conjecture (due to Keller) asserts the following.

Conjecture. (Jacobian Conjecture in dimension two) If the Jacobian of the pair (f, g) is a non-zero constant, then the map $(x, y) \mapsto$

The research of the author was supported in part by the NSA grants H98230-08-1-0129, H98230-06-1-0034 and H98230-11-1-0148.

$(f(x, y), g(x, y))$ is invertible. Note that the opposite is clearly true, because the Jacobian of any polynomial map is a polynomial, and, when the map is invertible, it must have no zeroes, so it is a constant.

The Jacobian Conjecture and its generalizations received considerable attention in the past, see for example [1], [7]. It is notorious for its subtlety, having produced a substantial number of incorrect proofs by respectable mathematicians.

From the point of view of a birational geometer, the most natural approach to the two-dimensional Jacobian Conjecture is the following. Suppose a counterexample exists. It gives a rational map from P^2 to P^2 . After a sequence of blow-ups of points, we can get a surface X with two maps: $\pi : X \rightarrow P^2$ (projection onto the origin P^2) and $\varphi : X \rightarrow P^2$ (the lift of the original rational map).

Note that X contains a Zariski open subset isomorphic to A^2 and its complement, $\pi^*((\infty))$, is a tree of smooth rational curves. We will call these curves exceptional, or curves at infinity. The structure of this tree is easy to understand inductively, as it is built from a single curve (∞) on P^2 by a sequence of two operations: blowing up a point on one of the curves or blowing up a point of intersection of two curves. However, a non-inductive description is probably impossible, which is the first difficulty in this approach. Another difficulty comes from the fact that the exceptional curves on X may behave very differently with respect to the map φ . More precisely, there are four types of curves E .

- type 1) $\varphi(E) = (\infty)$
- type 2) $\varphi(E)$ is a point on (∞)
- type 3) $\varphi(E)$ is a curve, different from (∞)
- type 4) $\varphi(E)$ is a point not on (∞)

From a first glance, the situation appears almost hopeless. One of the goals of this paper is to show that it is really not that bad. In particular, for a given graph of curves, one can essentially always tell which curves are of which type, and there is a fairly restrictive family of graphs that can potentially appear in a counterexample to the Jacobian Conjecture. Our main tools are the basic tools of algebraic geometry of surfaces: the intersection pairing and the adjunction formula. We are guided by some of the ideas of the log Minimal Model Program, but most of the proofs are relatively elementary and self-contained.

This paper is organized as follows. Section 2 is devoted to preliminary results on the graph of the exceptional curves. In section 3 we use slightly more subtle arguments to get further restrictions on that graph and the structure of the Stein factorization of the morphism φ . Section 4 is devoted to adjunction formulas and inequalities that generalize the arguments of section 3 to more general compactifications of the target

plane. In section 5 we develop and use the theory of the determinants of weighted trees and forests and get some further restrictions on the structure of the exceptional graph. In section 6 we try to attack the problem by exploring different compactifications of the target plane. Finally, section 7 contains some informal discussion of possible further developments of our approach.

Acknowledgments. This paper is dedicated to the memory of V.A. Iskovskikh, who introduced the author to the beauty of birational geometry. The author is also indebted to David Wright and Ed Formanek for stimulating discussions related to the Jacobian Conjecture.

2. PRELIMINARY OBSERVATIONS AND DEFINITIONS

We change the notation slightly from the Introduction.

Suppose $X = P^2$, $Y = P^2$ and $\varphi_Y^X : X \dashrightarrow Y$ is a rational map. Suppose further that on an open subset $A^2 \subset P^2 = X$ the map φ_Y^X is defined, unramified, and $\varphi_Y^X(A^2) \subseteq A^2 \subset P^2 = Y$. By a sequence of blow-ups at smooth points, we get a surface Z with a birational map $\pi : Z \rightarrow X$ and a generically finite map $\varphi_Y^Z : Z \rightarrow Y$ such that $\varphi_Y^Z = \varphi_Y^X \circ \pi$. We will denote $\varphi = \varphi_Y^Z$.

The blow-ups that lead to Z can be done outside of $A^2 \subset X$. So $Z = A^2 \cup (\cup E_i)$, where E_i are rational curves. The following proposition collects some straightforward observations.

Proposition 2.1. *1) The curves E_i form a tree.*

2) One of E_i is $\pi^{-1}(\infty)$, all others are mapped to points by π .

3) The classes of E_i form a basis of the Picard group of Z .

The structure of Z is to a large extent determined by the graph of intersections of E_i . The vertices of this graph correspond to E_i -s and are usually labeled by E_i^2 . The edges correspond to the points of intersections of two different E_i -s. The graph is a tree.

This graph is not so easy to deal with because blowing up a point changes the self-intersections of the curves passing through it. Inspired by the Minimal Model Program, we consider a different labeling of this graph. We consider the augmented canonical class of Z , $\bar{K}_Z = K_Z + \sum_i E_i$. It can be uniquely written as a linear combination of E_i , $\bar{K}_Z = \sum_i a_i E_i$. We label the vertices of the intersection graph by these numbers a_i .

With this labeling we now describe what happens when a point is blown up, in any of the intermediate steps in getting from X to Z .

Proposition 2.2. *When a point is blown up, going from Z' to Z'' , one of the following two operations is performed to the graph of the exceptional curves:*

1) *A new vertex is added to the graph, connected to one of the vertices. It is labeled a_i+1 , where a_i is the label of the vertex it is connected to.*

2) *A new vertex is introduced on the edge connecting two vertices, “breaking” the edge into two edges. The new vertex gets labeled with $a_i + a_j$, where a_i and a_j are the labels of the two vertices it is connected to.*

Proof. The first case corresponds to blowing up a point on one of the curves. The second case corresponds to blowing up an intersection of two curves. The augmented canonical class calculations are straightforward and are left to the reader. \square

Notice that once a vertex is created, its label never changes, which is in sharp contrast with the traditional labeling.

The following observation is true for any Z , unrelated to the map φ . It is easily proven by induction on the number of exceptional curves, using the above proposition.

Proposition 2.3. *For any two adjacent vertices E_i, E_j of the graph of Z , $\gcd(a_i, a_j) = 1$. In particular, no two adjacent vertices have even labels.*

The following example serves two purposes. It shows how the graph of Z is constructed from the graph of $X = P^2$, and we will use it to compare our labeling with the traditional self-intersection labeling.

Example.

We start with $X = P^2$, its graph is the single vertex.

$$\begin{array}{c} \circ \\ -2 \end{array}$$

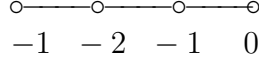
Blowing up a point, we get

$$\begin{array}{c} \circ \text{---} \circ \\ -1 \quad -2 \end{array}$$

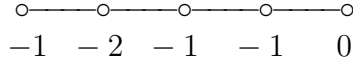
Blowing up another point on the pullback of (∞) , we get

$$\begin{array}{c} \circ \text{---} \circ \text{---} \circ \\ -1 \quad -2 \quad -1 \end{array}$$

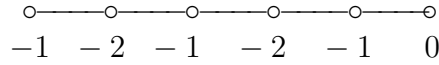
Blowing up a point on a newly blown up curve, we get



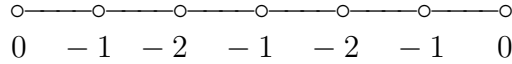
Then we blow up the intersection of the last two curves and get



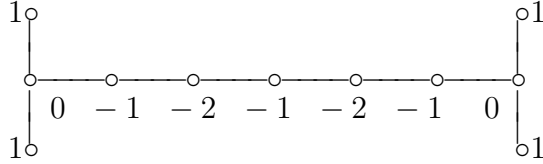
Blowing up another intersection point, we get



Blowing up another point, we get the following graph



Finally, blowing up four more points (in any order) we get the following:



For most of the exceptional curves, one can easily recover their self-intersection from the graph, using the adjunction formula:

$$(K_Z + E_i)E_i = -2, \text{ so } \bar{K}_Z \cdot E_i = -2 + \#(E_j \text{ adjacent to } E_i)$$

Thus, if $\bar{K}_Z = \sum a_i E_i$, we have

$$a_i E_i^2 + \sum_{E_j \text{ adj. } E_i} a_j = -2 + \#(E_j \text{ adjacent to } E_i)$$

So if $a_i \neq 0$, E_i^2 can be easily calculated.

However, when $a_i = 0$, it is not that easy. One can see in the above example, the left curve with $a_i = 0$ has self-intersection (-3) , while the right one has self-intersection (-4) , despite the symmetry of the graph. One can remedy this situation by keeping track of the strict pullback of infinity. We do not need it in this paper, and the details are left to an interested reader.

Note that the subgraph of vertices with negative labels is connected. It is separated from the “positive” vertices by the “zero” vertices. Moreover, the “zero” vertices are only connected to vertices with labels (-1) or 1 .

Now we are going to make use of the map φ . The main idea is to use the adjunction formula for φ to get a formula for \bar{K}_Z .

Recall from the Introduction the four types of curves E_i . For every curve of type 1 or 3 denote by f_i the degree of the map onto its image and by r_i the ramification index. Denote by L the class of the line on $Y = P^2$.

Proposition 2.4. *There exist integers b_i for the curves E_i of types 2 and 4 such that*

$$\bar{K}_Z = \varphi^*(-2L) + \sum_{\text{type}(E_i)=3} r_i E_i + \sum_{\text{type}(E_i)=2 \text{ or } 4} b_i E_i$$

Proof. Consider the differential 2-form ω on $Y = P^2$ that has the pole of order 3 at (∞) and no other poles or zeroes. Because φ is unramified on the $A^2 \subset X$, there is a differential form on Z , such that its divisor of zeroes and poles is $\varphi^*(-3L) + \sum_i c_i E_i$, where c_i can be calculated locally at a general point of each E_i .

Notice that for the curves E_i of types 1 and 3, $c_i = r_i - 1$, and

$$\varphi^*(L) = \sum_{\text{type}(E_i)=1} r_i E_i + \sum_{\text{type}(E_i)=2} e_i E_i$$

for some e_i . Thus,

$$\begin{aligned} \bar{K}_Z &= K_Z + \sum E_i = \varphi^*(-3L) + \sum_{\text{type}(E_i)=1 \text{ or } 3} r_i E_i + \sum_{\text{type}(E_i)=2 \text{ or } 4} (c_i + 1) E_i = \\ &= \varphi^*(-2L) + \sum_{\text{type}(E_i)=3} r_i E_i + \sum_{\text{type}(E_i)=2 \text{ or } 4} b_i E_i \end{aligned}$$

□

Note that because E_i are independent in the Picard group of Z , the above representation of \bar{K}_Z is unique and must match with the labeling of the graph of E_i . As a corollary, we have the following observation.

Proposition 2.5. *1) Any curve of type 1 has a negative even label.*

2) Any curve of type 3 has a positive label.

Proof. Note that $\varphi^*(-2L)$ only involves curves of type 1 and 2. □

Additionally, the union of curves of type 1 and 2 must be connected, as a specialization (set-theoretically) of a pullback of a generic L on $Y = P^2$. This means that the corresponding subgraph is connected.

Every curve of type 3 must intersect with one of the curves of type 1 or 2, while the curves of type 4 do not intersect with curves of type 1 or 2. (This follows from the projection formula of the intersection theory: if E is a curve on Z , $E \cdot \varphi^*(L) = (\varphi_* E) \cdot L$.)

On the other hand, a type 3 curve cannot intersect a type 1 curve, because negative and positive labels are never adjacent. Because the

graph of the exceptional curves on Z is a tree, no two curves of type 3 intersect with each other. Putting this all together, we must have the following. The tree of curves on Z has a connected subtree containing all curves of type 1 and 2. Some of the vertices of this subtree may have one or more curves of type 3 connected to them. Then some of these type 3 curves may have trees of type 4 curves connected to them. Additionally, no two curves of type 1 are adjacent, and the subtree of curves of type 1 and 2 contains the connected subtree of curves with negative labels.

Proposition 2.6. $\pi^{-1}(\infty)$ is of type 1 or 2.

Proof. One can prove it using the above description of the graph of exceptional curves, but there is also the following direct geometric argument. The pullbacks of lines on $X = P^2$ form a family of rational curves C on Z that intersect $\pi^{-1}(\infty)$ at a generic point and do not intersect any other exceptional curves. Consider $\varphi(C)$ for a generic C . If $\pi^{-1}(\infty)$ is of type 3 or 4 then $\varphi(C) \subseteq A^2 \subset Y$. The curve C is proper and A^2 is affine, so $\varphi(C)$ is a point, which is impossible. \square

Until now, the variety Z was an arbitrary resolution at infinity of the original rational map. But we can put an additional restriction on it, to avoid unnecessary blow-ups.

Definition 2.1. *If a curve is obtained by blowing up the intersection of two curves, we call these curves its parents. If a curve is obtained by blowing up a point on one of the curves, this curve is called its parent. The original line at infinity has no parents. Note that other curves may be created afterwards that separate the curve from one or both of its parents.*

Definition 2.2. *For a given curve E , the set of its ancestors $A(E)$ is the smallest set S of the exceptional curves that contains its parent(s) and has the property that it contains the parents of every curve in S . Note that this set is empty if E is the original line at infinity. Otherwise, it consists of the original line at infinity and all curves that have to be created before E .*

Definition 2.3. *A curve E_i on Z is called **final** if there is a sequence of blow-ups from X to Z such that E_i is blown up last. Equivalently, a curve is final if it is not a parent to any curve.*

Note that there may be more than one final curve, and $\pi^{-1}(\infty)$ is never final. In what follows, E_i is one of the exceptional curves on Z .

Proposition 2.7. *Suppose that when Z was created, E_i was created after all of its neighbors in the graph (i.e. all adjacent vertices). Then E_i is a final curve.*

Proof. Instead of creating E_i at its due time we can change the order of blow-ups and create it at the last step of the process, without changing anything else. \square

Proposition 2.8. *Suppose $a_i = a(E_i) \geq 2$ and it is the largest label among all its neighbors. Then E_i is final.*

Proof. We will prove that E_i was created after all its neighbors. First of all, no neighbor of E_i can be a blow-up of a point on E_i , because its label would have been $a_i + 1$. If it were a blow-up of a point of intersection of E_i and some E_j , then E_i and E_j were adjacent before the blow-up. Negative curves are never adjacent to the positive curves and zero curves are only adjacent to curves with labels 1 or -1 . Thus, $a_j \geq 1$. So the label of the new curve is $a_i + a_j \geq a_i + 1 > a_i$. \square

Note that no two curves with the same label $a_i \geq 2$ can be adjacent, by Proposition 3. So every local maximum $a_i \geq 2$ is a strict maximum.

Proposition 2.9. *If $a_i = 1$, then E_i is final if and only if it either has only one neighbor, with label 0, or exactly two neighbors, with labels 1 and 0.*

Proof. A curve with label 1 can be created either by a blow-up of a point on a curve with label 0 or by a blow-up of an intersection of a curve with label 0 and a curve with label 1. Once created, it will be final if and only if no other curve is blown up as its neighbor. The rest is easy and is left to the reader. \square

The above two propositions allow us to easily spot the final curves in the positive part of the graph of curves. Our interest in the final curves stems from the following. If one of the final curves on Z is of type 2 or 4, then it can be contracted, using the φ -relative MMP to get another Z , with two maps to X and Y and a smaller Picard number.

Definition 2.4. *We call Z **minimal** if all of its final curves are of type 1 or 3.*

Proposition 2.10. *If a counterexample to the Jacobian Conjecture exists, it can be obtained using a minimal Z .*

Proof. Take Z with smallest possible Picard number. If it is not minimal, it can be created in such a way so that some curve of type 2 or 4 is blown-up last. Using MMP relative to φ , it can be blown down, maintaining the morphisms, and creating a counterexample to the Jacobian Conjecture with smaller Picard number. \square

From now on, Z will always be minimal.

Proposition 2.11. *Suppose E is a curve of type 3 on Z . Suppose E_0 is the curve of type 2 it is adjacent to. Then the tree on the other side of E is a line $E - E_1 - \dots - E_k$, where E_1, \dots, E_k are of type 4.*

Proof. The label of E is positive. All curves E_1, \dots, E_k “on the other side” of E are of type 4. They must be ancestors of some curve of type 3, so they are all ancestors of E . If the connected component of the graph obtained from Γ by removing E is not a line, there would have to be another final curve there, which is impossible. \square

3. OTHER VARIETIES AND FURTHER ANALYSIS

We start with the theorem that shows that type 3 curves must exist in a counterexample to the Jacobian Conjecture. Note that the type 3 curves are called “di-critical components” in [5], [6], and this fact is well known and can be easily proven by a topological argument. So the main purpose of our proof is to show an easy application of our method before proceeding to the more intricate questions.

Theorem 3.1. *Suppose Z and φ provide a counterexample to the Jacobian Conjecture. Then Z contains a curve of type 3, where φ is ramified.*

Proof. Consider a generic line L on the target variety $Y = P^2$. The curve $C = \varphi^{-1}(L)$ is smooth and irreducible (“Bertini’s theorem”). Moreover, we can assume that for all but finitely many lines L' that only intersect L “at infinity”, $C' = \varphi^{-1}(L')$ is smooth and irreducible. We can also assume that L does not pass through the images of the exceptional curves of types 2 and 4, so C does not intersect these curves on Z . Suppose that the genus of C is g , the map $H = \varphi|_C C \rightarrow L$ has degree n and the number of points of C “at infinity” is k . (There is a special point ∞ on L , the only one not lying in A^2 . The number k is the number of points of C mapped to it, in a set-theoretic sense.) Because the map φ is only ramified at the exceptional curves of X , the map H could only be ramified at these k points at infinity. By Hurwitz formula, we have

$$2g - 2 = -2n + r,$$

where r is the total ramification at infinity. We have $g \geq 0$, $n \geq 1$ and $r \leq n - k$. So

$$-2 \leq 2g - 2 \leq -2n + n - k = -n - k \leq -2$$

Thus all the inequalities above are equalities, $g = 0$, $n = 1$, and $k = 1$. Because $n = 1$, the map φ is birational. For the birational maps the Jacobian Conjecture is well known (see, e.g. [1]). \square

Now we want to make further use of the morphism $\varphi : Z \rightarrow Y$. We decompose it into a composition of two morphisms, birational and finite (Stein factorization):

$$Z \longrightarrow W \longrightarrow Y$$

Here the first morphism is birational and denoted τ , and the second one is finite and denoted ρ .

The surface W is normal. In what follows, we use the intersection theory for normal surfaces due to Mumford. Suppose K_W is its canonical class, as the Weil divisor class modulo numerical equivalence. We define the augmented canonical class $\bar{K}_W = K_W + \sum E_i$.

Proposition 3.1. *In the situation and notation described above,*

$$\bar{K}_W = \rho^*(-2L) + \sum_{\text{type}(E_i)=3} r_i E_i.$$

Proof. The curves E_i on W are exactly the images of curves of types 1 and 3 on Z . By adjunction, we have:

$$K_W = \rho^* K_Y + \sum (r_i - 1) E_i,$$

where r_i is the ramification index, and E_i are the images of the curves E_i of types 1 and 3 on Z .

$$\bar{K}_W = \rho^*(-3L) + \sum r_i E_i = \rho^*(-2L) + \sum_{\text{type}(E_i)=3} r_i E_i.$$

\square

Note that if one denotes by \bar{K}_Y the class of $K_Y + (\infty)$, then $\rho^*(-2L)$ in the above Proposition is $\rho^*(\bar{K}_Y)$. The next theorem is very important. It will be further strengthened at the end of this section.

Theorem 3.2. *(Big Ramification Theorem)*

Suppose Z is a counterexample to the Jacobian Conjecture. Then on W the “di-critical log-ramification divisor”

$$\bar{R} = \sum_{E_i \subset W, \text{type}(E_i)=3} r_i E_i$$

intersects positively with all exceptional curves of type 3. As a corollary, $\bar{R}^2 > 0$.

Proof. Suppose that $C = E_i$ is a curve of type 3 on W . Suppose d_i is the degree of $\rho(C)$. Suppose $\tau : Y_2 \rightarrow W$ is a minimal resolution of singularities of W . Then

$$\bar{K}_W E_i = (K_W + E_i)E_i + \sum_{j \neq i} E_i E_j,$$

where E_j are curves of type 1 or 3. Note that because E_i intersects at least one curve of type 1, $\bar{K}_W E_i > (K_W + E_i)E_i$. Lifting up to Y_2 , we get

$$(K_W + E_i)E_i = (\tau^*(K_W) + \tau^*(E_i))\tau^{-1}(E_i) \geq (K_{Y_2} + \tau^{-1}(E_i))\tau^{-1}(E_i) \geq -2.$$

So for all i $\bar{K}_W E_i > -2$.

Therefore,

$$\bar{R} \cdot E_i = 2\varphi^*(L) \cdot E_i + \bar{K}_W \cdot E_i > 2f_i d_i - 2 \geq 0$$

□

Corollary 3.1. *The curve $\pi^{-1}(\infty)$ is of type 2.*

Proof. By Proposition 2.6, it is of type 1 or 2. If it is of type 1, then it is not included in $\sum_{type(E_i)=3} r_i \tau^* E_i$. So $\sum_{type(E_i)=3} r_i \tau^* E_i$ consists of curves contractible by π . But

$$\left(\sum_{type(E_i)=3} r_i \tau^* E_i \right)^2 = \left(\sum_{type(E_i)=3} r_i E_i \right)^2 > 0,$$

contradiction. □

Note that every curve of type 3 on Z intersects the union of curves of type 2 at exactly one point, and does not intersect curves of type 1. When the curves of type 2 are contracted, on W , every curve of type 3 intersects the union of curves of type 1 at exactly one point.

Proposition 3.2. *For every curve E_i of type 3 on W the point above is $\tau(\pi^{-1}(\infty))$.*

Proof. Suppose there is a point $w \in W$ on the union of type 1 curves, which is not $\tau(\pi^*(\infty))$ and which has some type 3 curves passing through it. Define

$$\bar{R}_w = \sum_{w \in E_i, type(E_i)=3} r_i E_i$$

Because the curves of type 3 not passing through y cannot intersect any components of \bar{R}_w , we have $\bar{R}_w^2 = \bar{R}_w \cdot \bar{R}$. By Theorem 3.2, this implies that $\bar{R}_w^2 > 0$. Like in the proof of Corollary 3.1, $\tau^*(\bar{R}_w)$ consists of curves contractible by π , which is impossible. □

Proposition 3.3. *On W , all exceptional curves contain $\tau(\pi^{-1}(\infty))$ and there are no other points of intersection.*

Proof. By the proposition above, every curve of type 3 contains $\tau(\pi^{-1}(\infty))$ and this is its only point of intersection with other exceptional curves. Now consider a curve E_i of type 1. Suppose it does not contain $\tau(\pi^{-1}(\infty))$. Then it does not intersect any of the curves of type 3 on W .

On W we have:

$$\bar{K}_W \cdot E_i \geq (K_W + E_i)E_i \geq -2$$

On the other hand,

$$\bar{K}_W \cdot E_i = (-2\rho^*(L) + \bar{R}) \cdot E_i = -2\rho^*(L) \cdot E_i \leq -2$$

The inequalities above become equalities only if E_i intersects no other curves and is smooth. This would make it the only curve of type 1 on W , which would then have to intersect with some curves of type 3, contradiction. \square

Thus, we know that every curve of type 1 on W contains $\tau(\pi^{-1}(\infty))$. We now look at the graph of curves on Z . The curves of type 2 that are mapped to $\tau(\pi^{-1}(\infty))$ form a connected subgraph, containing $\pi^{-1}(\infty)$. Every curve of type 1 or 3 is attached to this subgraph. On “the other side” of each curve of type 3 there may be a single chain of curves of type 4, and on “the other side” of each curve of type 1 there may be a single chain of curves of type 2. Note that all of these “other side” curves must be created before the corresponding type 3 or type 1 curves. When mapped to W , the curves of type 1 and 3 intersect at $\tau(\pi^{-1}(\infty))$ and nowhere else.

One can restrict the structure of the possible counterexamples even further.

Theorem 3.3. *In any counterexample to the Jacobian Conjecture there are no curves on “the other side” of the curves of type 1.*

Proof. Consider a curve of type 1, E , on Z . Suppose the ramification index at E is r . Then the coefficient of $\varphi^*(L)$ in E is r , and the coefficient of \bar{K}_Z is $(-2r)$. Consider the divisor class $D = \bar{K}_Z + 2\varphi^*(L) = \dots + 0 \cdot E + x_1 E_1 + \dots + x_k E_k$, where E_1, \dots, E_k are the curves on Z “on the other side” of E . We know that D intersects by zero with E_1, \dots, E_{k-1} . It intersects by -1 with E_k . We formally add another vertex to the graph, “ E_{k+1} ” and set the coefficient of D at it to be 1. (Note that we are not blowing up any points and E_{k+1} does not have any geometric meaning). We now have a chain $E, E_1, \dots, E_k, E_{k+1}$

and a divisor $D' = 0 \cdot E + x_1 E_1 + \dots + x_k E_k + 1 \cdot E_{k+1}$, such that D' intersects by zero with all E_1, E_2, \dots, E_k . Because the self-intersections of all E_i , $1 \leq i \leq k$, are less than or equal to -2 , the coefficients x_i must form a concave up chain between 0 and 1, contradicting their integrality. (Here is a more formal argument. Suppose at least one of the x_i , $1 \leq i \leq k$, is not positive. Then consider the minimum of x_i , obtained at x_j , such that $x_{j+1} > x_j$, where formally $x_0 = 0, x_{k+1} = 1$. Then $D' \cdot E_j \geq x_{j-1} + x_{j+1} + 2x_j > 0$, contradiction. Suppose the maximum of x_i , $1 \leq i \leq k$, is greater than or equal 1 and is obtained at x_j , where $x_{j-1} < x_j$. Then $D' \cdot E_j \leq x_{j-1} + x_{j+1} - 2x_j < 0$, contradiction. Thus all x_i are strictly between 0 and 1, which is impossible because they are integers.) \square

As a corollary of these observations, we get a rather detailed description of the structure of W .

Theorem 3.4. *Suppose W is defined as above for a counterexample to the Jacobian Conjecture, E_i are images of the exceptional curves on it. Then $W \setminus \cup_i E_i$ is isomorphic to the affine plane. There exist distinct points $A = \tau(\pi^{-1}(\infty))$ and $A_i \in E_i$ (at most one for each E_i of type 3) so that W is smooth outside of them, A is a normal singularity, A_i are cyclic quotient singularities, all the curves E_i pass through A , do not intersect elsewhere and are smooth in the nonsingular part of W . For all exceptional curves E_i that do not contain A_i , $E_i \setminus \{A\}$ is isomorphic to the affine line. For all curves E_i that contain A_i , $E_i \setminus \{A, A_i\}$ is isomorphic to the algebraic torus (affine line with a removed point).*

Proof. Most of the statements have already been proven. To finish the proof, note the following. The map τ from Z to W contracts all curves of types 2 and 4, and no curves of types 1 and 3. The curves of type 2 form a subtree on Z , so they are contracted to one singular point A . Note that all curves of type 1 and 3 on Z intersect this subtree at exactly one point. Some curves of type 3 have one chain of curves of type 4 attached to them, that get contracted into a cyclic quotient singularity. \square

Note that the above theorem restricts greatly the restriction of the map φ to the exceptional curves of types 1 and 3.

Theorem 3.5. 1) For all curves E_i of type 1, $f_i = 1$.

2) For all curves E_i of type 3 either $f_i = 1$ or the restriction to E_i of the map from Z to Y is isomorphic to the composition of a map $(x \mapsto x^{f_i}) : P^1 \rightarrow P^1$ and a generically one-to-one map from P^1 to a possibly singular rational curve (the normalization map for $\varphi(E_i)$).

Proof. This follows from the fact that the restriction to E_i of the map from Z to Y can only be ramified at the points of intersections of E_i and other exceptional curves, and the classification of self-maps of the projective line that are ramified at one or two points. \square

For the following theorem, we need to introduce additional notation.

Definition 3.1. Suppose Z, W, Y and $\varphi : W \rightarrow Y$ are as above. For a type 3 curve E_i , denote by F_i its image on Y , as a reduced irreducible divisor (a possibly singular rational curve). Then

$$\rho^*(F_i) = r_i E_i + G_i,$$

where G_i is an effective Weil divisor. Its irreducible components are curves in A^2 that are mapped to F_i . We will call these curves *coexceptional*.

Theorem 3.6. Suppose $E_i \subset W$ is a type 3 curve that contains a cyclic quotient singularity A_i . Then some coexceptional curve from G_i contains A_i .

Proof. The proof is very similar to the proof of Theorem 3.3. Suppose the support of G_i does not contain A_i . To simplify the notation, denote E_i by E ; suppose E_1, E_2, \dots, E_k are the curves of type 4 that are mapped to A_i , with E_1 intersecting E . Suppose $\varphi : Z \rightarrow Y$ is our map. Consider on Z the divisor $D = \bar{K}_Z - \varphi^*(F_i)$. We can write D as a linear combination of exceptional curves and the strict pullbacks of the coexceptional curves. Because the coefficient of E in this linear combination is zero, and the only curves that contribute to the intersection of D with E_1, \dots, E_k are E_1, \dots, E_k , we get a linear combination with integer coefficients $x_1 E_1 + \dots + x_k E_k$ that intersects by zero with E_1, \dots, E_{k-1} and by (-1) with E_k . We can now follow verbatim the argument in Theorem 3.3 to get a contradiction. \square

We end this section with a strengthening of Theorem 3.2.

Theorem 3.7. (*Ample Ramification Theorem*)

Suppose Z is a counterexample to the Jacobian Conjecture. Then on W the “di-critical log-ramification divisor”

$$\bar{R} = \sum_{E_i \subset W, \text{type}(E_i)=3} r_i E_i$$

is ample.

Proof. The surface W is rational and, therefore, \mathbb{Q} -factorial. So the Weil divisor \bar{R} is \mathbb{Q} -Cartier. It is effective and by Theorem 3.2 it intersects positively with all of its irreducible components. So by the

Nakai-Moishezon criterion it is enough to show that it intersects positively with all irreducible curves C on W that are not the exceptional curves of type 3.

If C is a curve of type 1 on W , then it intersects the support of \bar{R} at $A = \varphi(\pi^{-1}(\infty))$, so $C \cdot \bar{R} > 0$.

If C is any other curve on W , that does not intersect positively with \bar{R} , it does not intersect with any curves of type 3 and it must intersect with at least one curve of type 1 (because it cannot be contained entirely in the affine plane). By the Hodge Index Theorem and Theorem 3.2, $C^2 < 0$. Note also that C does not pass through A , so it intersects with at least one curve of type 1 at a smooth point. Because the \bar{K} -labels of all curves of type 1 are at most -2 , $K_W \cdot C \leq -3$. Therefore, $(K_W + C) \cdot C \leq -3 < -2$, which is impossible. \square

4. ADJUNCTION FORMULAS AND INEQUALITIES

As the above discussions clearly indicate, one of the basic objects that we need to consider are algebraic surfaces with a collection of reduced irreducible curves on them. The natural framework for this is provided by the notion of varieties with boundaries. They have been studied extensively as part of the Minimal Model Program, and most of our results in this section are not new, and hold in a much more general setting. A good introduction to the more general theory can be found in [9]. We start by recalling several standard definitions, but this section is directed at the readers with some background in modern birational geometry. All varieties are over \mathbb{C} , all surfaces are normal unless otherwise specified and \mathbb{Q} -factorial whenever needed. Please note that the notation in the beginning of this section is independent of the notation in the rest of the paper.

Definition 4.1. *A log-surface (or surface with boundary) is a pair (X, D) where X is an algebraic surface, and $D = \sum_i D_i$ is a sum of prime divisors (i.e. reduced irreducible curves) on X . The augmented canonical class of (X, D) is $\bar{K} = K + D$, where K is the canonical class of X . The divisor D is called the boundary, and D_i are the components of the boundary. We will often omit D from the notation, when it is clear from the context what it is (usually, a complement of A^2 in X).*

Definition 4.2. *A resolution of singularities of a log-surface (X, D) is a smooth algebraic surface Y with a birational map $\tau : Y \rightarrow X$ such that the exceptional divisors (curves) of τ and the components of $\tau^{-1}(D)$ have simple normal crossings (i.e. all the irreducible components are smooth and intersect transversally).*

Note that whenever we have a birational map $\tau : Y \rightarrow X$, we take for the boundary on Y the union of the components of $\tau^{-1}(D)$ and the exceptional divisors. A resolution as above always exists, moreover it can be obtained by a sequence of blow-ups of the surface or the irreducible components of the boundary or the points where the components of the boundary do not intersect transversally.

Definition 4.3. *For a birational map $\tau : Y \rightarrow X$, the adjunction formula is an equality of \mathbb{Q} -Cartier divisors*

$$\bar{K}_Y = \tau^* \bar{K}_X + \sum_i a_i E_i,$$

where the sum is over the exceptional divisors of τ and a_i are rational numbers called log-discrepancies.

Definition 4.4. *A log-surface (X, D) has log-canonical singularities iff for one (any) resolution of singularities Y as above all log-discrepancies are nonnegative.*

Example 4.1. *If X is smooth and D has simple normal crossings, then (X, D) has log-canonical singularities.*

Example 4.2. *Suppose a point $P \in X$ is a normal surface singularity and D_1 and D_2 are curves on X containing P . Suppose the dual graph of exceptional curves on the minimal resolution $\pi : \tilde{X} \rightarrow X$ of X consists of a chain of rational curves E_1, \dots, E_k (i.e. all E_i are isomorphic to P^1 and E_i intersects E_{i+1} transversally for $1 \leq i \leq k-1$ with no other points of intersection). Suppose that the strict pullback of D_1 intersects E_1 transversally and the strict pullback of D_2 intersects E_k transversally, so that $\pi^{-1}D_1, E_1, \dots, E_k, \pi^{-1}D_2$ is a chain of rational curves. Denote $D = D_1 + D_2$. Then (X, D) is log-canonical in the neighborhood of P . (Note that in this case the singularity of X at P is locally-analytically a cyclic quotient). Note that in this case the resolution $\tau : Y \rightarrow X$ is log-crepant (i.e. all log-discrepancies are zero).*

The following theorem is the result of the relative Minimal Model Program.

Theorem 4.1. *Suppose (S, D) is a log-surface, and $\tau : V \rightarrow S$ is its resolution of singularities. Then the map τ can be decomposed into a composition of two maps, $\chi : V \rightarrow S_1$ and $\mu : S_1 \rightarrow S$, such that*

- 1) *The log-surface (S_1, D_1) has log-canonical singularities;*
- 2) *The augmented canonical class \bar{K}_{S_1} is μ -ample (i.e. $\bar{K}_{S_1} E_i > 0$ for all E_i that are contracted to a point by μ).*

Here D_1 is, as usual, the union of components of $\mu^{-1}D$ and the exceptional divisors of μ . This S_1 with the morphism μ is called the *log-canonical modification* of S .

Proof. This can be done by contracting one-by-one exceptional curves of τ that have negative intersection with the augmented canonical class. See, e.g. the paper of Fujino [8], section 3 for this theory in a much more general setting. \square

Because of the above theorem, it is important to classify all log-canonical singularities with boundary coefficients 1. The general classification of log-canonical surface singularities is well known, see the work of Alexeev [3]. We are most interested in the surface being log-canonical along the boundary D , especially at the points of intersections of components of D .

Theorem 4.2. *Suppose $P \in X$ is a point of intersection of two components of D . Then P is a log-canonical singularity if and only if it is of the kind described in the Example 4.1 or 4.2.*

Proof. This follows from the classification of [3], and can also be obtained independently. \square

We now proceed to applications, and need to introduce a new definition.

Definition 4.5. *Suppose (S, D) is log-surface. Suppose D_i is an irreducible component of D . Then the valency of D_i is the number of points in the normalization of D_i that are mapped by the normalization map to the points of intersection of D_i and some other D_j . We denote the valency of D_i by $\text{val}_D(D_i)$ or simply $\text{val}(D_i)$ if D is clear from the context.*

The notion of valency generalizes to the singular surfaces the notion of the number of neighbors in the dual graph of exceptional curves. The following theorem follows immediately from the adjunction formula for the curve D_i on the surface S .

Theorem 4.3. *Suppose S is a smooth surface, and the boundary D is a union of not necessarily smooth curves intersecting transversally at smooth points (i.e. D has normal crossings but not necessarily simple normal crossings). Suppose D_i is one of the irreducible components of D . As usual, \bar{K}_S stands for $K_S + D$. Then*

$$\bar{K}_S \cdot D_i \geq (2g - 2) + \text{val}(D_i),$$

and the equality occurs if and only if D_i is smooth.

The following theorem is an appropriate generalization of the above to arbitrary surfaces with boundaries.

Theorem 4.4. (*Adjunction Inequality*) *Suppose (S, D) is a log-surface, D_i is an irreducible component of the boundary. Suppose the genus of the normalization of D_i is g . Then*

$$\bar{K}_S \cdot D_i \geq (2g - 2) + \text{val}(D_i)$$

Moreover the equality occurs if and only if all of the following conditions are satisfied:

- 1) (S, D) is log-canonical in the neighborhood of D_i ;
- 2) S and D_i are smooth at all points P on D_i that do not lie on other boundary components.

Proof. Suppose (S', D') is the log-canonical modification of (S, D) , with the morphism $\mu : S' \rightarrow S$. Because $\bar{K}_{S'}$ is μ -ample, all log-discrepancies a_i in the adjunction formula

$$\bar{K}_{S'} = \mu^* \bar{K}_S + \sum_i a_i E_i$$

are positive. (Note that if (S, D) is log-canonical, then $S' = S$ and then there are no exceptional curves E_i).

Denote $D'_i = \mu^{-1} D_i$. Note that if S' is not equal to S in the neighborhood of D_i , then

$$\bar{K}_{S'} \cdot D'_i < \bar{K}_S \cdot D_i$$

It is also easy to see that $\text{val}(D'_i) = \text{val}(D_i)$. So to prove the desired inequality for D_i it is enough to prove it for D'_i .

Consider the minimal resolution of singularities $\chi : V \rightarrow S'$. Above every point of intersection of D_i with other boundary components we get a chain of rational curves with the end curves intersecting transversally with D_i . Define the boundary on V to be the union of $\chi^{-1} D'$ and these exceptional curves above the points of intersection of D'_i and other components of D' . Consider the adjunction formula:

$$\bar{K}_V = \chi^* \bar{K}_{S'} + \sum_j a_j E_j,$$

where E_j are the exceptional curves of χ .

Note that for the curves E_j that are mapped to the points of intersection of D'_i and other components of D' the log-discrepancies a_j are equal to zero. On the other hand, for all curves E_j that are mapped elsewhere on D'_i the discrepancies a_i are strictly negative, because $\chi^* K_{S'} - K_V$ is effective and $\chi^* D' - \chi^{-1} D'$ is a sum of E_j with strictly positive coefficients. Multiplying by $\chi^{-1} D'_i$, we get

$$\bar{K}_V \cdot \chi^{-1} D'_i \leq \bar{K}_{S'} \cdot D'_i$$

Applying Theorem 4.3 to V and $\chi^{-1}D'_i$, we get the desired inequality. Also, in order to have equality, several conditions need to be satisfied.

1) The surface S' must equal S in the neighborhood of D'_i , that is (S, D) is log-canonical in the neighborhood of D_i .

2) No exceptional curves are mapped by χ to a point on D'_i that does not lie on other components of the boundary.

3) The curve $\chi^{-1}D'$ must be smooth. By the previous conditions this implies that D is smooth outside of intersections with other boundary components.

Finally, it is easy to check that when these conditions are satisfied, all inequalities become equalities. \square

Definition 4.6. *If the boundary D on S satisfies the conditions of the above theorem for all irreducible components, we say that D has generalized simple normal crossings.*

Remark 4.1. *The number $(2g - 2) + \text{val}(D_i)$ is the degree of the augmented canonical divisor on the normalization of D_i , with the boundary being the pullback of the union of the other components of D by the normalization map.*

We now return our attention to a hypothetical counterexample to the Jacobian Conjecture. Generalizing the construction of previous sections, suppose Y is a compactification of the target A^2 , X a compactification of the origin A^2 ; $\pi : Z \rightarrow X$, $\varphi : Z \rightarrow Y$ is a minimal resolution of the rational map from X to Y and $Z \rightarrow W \rightarrow Y$ is the Stein factorization of φ , with the first map denoted by τ and the second one by ρ . Because φ is unramified on $\pi^{-1}(A^2)$, all the ramification of ρ occurs at the images by τ of the complement of A^2 on Z (i.e. the exceptional curves). All these surfaces will be equipped with the boundaries that are complements of A^2 that they naturally contain. The irreducible components of these boundaries will be called exceptional curves. Note that all exceptional curves are rational.

In this general setting we can still distinguish four classes of exceptional curves on Z . However, it is important to keep track for curves of type 1 what exceptional curve they are mapped to, and for curves of type 2 what exceptional curve(s) contain their image. Of most interest to us are the curves of type 3, i.e. the di-critical curves. Unlike the curves of type 1 and 2, they are essentially independent of the choices of the compactification. Just as before, the exceptional curves of type 2 and 4 get contracted by τ , so on W exceptional curves of types 1 and 3 remain. For each curve E_i of type 1 or 3 on W we will denote by F_i its image by μ . We denote by r_i the corresponding ramification index and by f_i the degree of the restriction of μ to E_i . We will often denote

the di-critical components by R_i to distinguish them from other exceptional curves. We will call $\bar{R} = \sum r_i R_i$ the di-critical log-ramification divisor. Its importance lies in the fact that it connects the augmented canonical classes of W and Y as follows.

Theorem 4.5. (*Adjunction Formula for ρ*) *Suppose W , Y and ρ are as above. Then*

$$\bar{K}_W = \rho^* \bar{K}_Y + \bar{R}$$

Proof. The argument is the same as in Proposition 3.1: because our polynomials map sends A^2 to A^2 , the preimage of every boundary component on Y is the union of the boundary components on W . The only ramification not accounted for in this way is in the di-critical components; since the boundary on W includes them, we get \bar{R} on the right hand side. The equality should be understood as the rational equivalence of Weil divisors. \square

In many cases, the surface Y will be smooth, with simple normal crossing boundary, but the argument works equally well when Y is a log-canonical surface with generalized simple normal crossing boundary. It is in this more general setting that we state the following theorem.

Theorem 4.6. *Under the above conditions, suppose $(Y, Y \setminus A^2)$ is a surface with generalized simple normal crossing boundary. Suppose E_i is an exceptional curve on W such that E_i does not intersect any di-critical curves on W . Then $(W, W \setminus A^2)$ has generalized simple normal crossings in the neighborhood of E_i .*

Proof. It is a standard matter to prove that W is log-canonical in the neighborhood of E_i . However we need a slightly stronger result, and will use Theorem 4.4. for it. Suppose $\rho(E_i) = F_i$, the ramification index of ρ at E_i is r_i and the degree of the restriction of ρ to E_i is f_i . Multiplying the adjunction formula for ρ by E_i , we get

$$\bar{K}_W \cdot E_i = f_i \cdot (\bar{K}_Y \cdot F_i)$$

By Theorem 4.4 and Remark 4.1, $\bar{K}_Y \cdot F_i$ is the degree of the augmented canonical divisor on the normalization of F_i . The adjunction formula for the map from the normalization of E_i to the normalization of F_i (the Riemann-Hurwitz formula) implies that $\bar{K}_W \cdot E_i$ is the degree of the augmented divisor on the normalization of E_i . (Note that the boundary there is exactly the pullback of the boundary on the normalization of F_i , and all ramification must occur there). Theorem 4.4. now implies the desired result. \square

We finish this section with some simple observations that describe explicitly the situation when the valency of F_i or E_i is small.

Theorem 4.7. *Under the conditions of the above theorem, the following is true.*

- a) *If $\text{val}(F_i) = 1$, then $\text{val}(E_i) = 1$ and $f_i = 1$.*
- b) *If $\text{val}(F_i) = 2$, then $\text{val}(E_i) = 2$. If $f_i > 1$, then the map from E_i to F_i is ramified at two points.*

Proof. As in the above theorem, $\bar{K}_W \cdot E_i = f_i \cdot (\bar{K}_Y \cdot F_i)$.

Note that E_i and F_i are rational, so the above theorem and Theorem 4.4. imply that $\text{val}(E_i) - 2 = f_i \cdot (\text{val}(F_i) - 2)$. This implies the result, considering that the map from E_i to F_i can only be ramified at the points of intersection with the other boundary components. \square

Theorem 4.8. *Under the conditions of Theorem 4.6, the following is true.*

- a) *If $\text{val}(E_i) = 1$, then $\text{val}(F_i) = 1$ and $f_i = 1$.*
- b) *If $\text{val}(E_i) = 2$, then $\text{val}(F_i) = 2$. If $f_i > 1$, then the map from E_i to F_i is ramified at two points.*
- c) *If $\text{val}(E_i) = 3$, then $\text{val}(F_i) = 3$ and $f_i = 1$.*

Proof. As in the proof of the previous theorem, $\text{val}(E_i) - 2 = f_i \cdot (\text{val}(F_i) - 2)$. The results follow immediately. \square

5. DETERMINANTS OF WEIGHTED FORESTS AND SOME APPLICATIONS

In this section we will construct and use a more sophisticated labeling on the graph of exceptional curves, using the determinants of the matrices related to the Gram matrix of the intersection form. This approach is related to the work of Domrina and Orevkov (cf. [5], [6]) and some of our results can be derived from theirs and/or the work of Walter Neumann ([12]). However our approach is slightly different: we do not use the splice diagrams and we derive applications by combining this labeling with the \bar{K} -labeling described above.

In what follows, a weighted tree means a connected graph with no cycles, with weights attached to the vertices. A weighted forest is a graph with no cycles, with weights attached to the vertices. To each such graph Γ and any ordering of the vertices we will associate a matrix $Q(\Gamma)$ as follows.

$$Q(\Gamma)_{i,j} = \begin{cases} -a_i, & \text{if } i = j, \\ -1, & \text{if } i^{\text{th}} \text{ and } j^{\text{th}} \text{ vertices are connected by an edge} \\ 0 & \text{otherwise} \end{cases}$$

In applications, the forest will be obtained from a tree of exceptional curves on Z by removing some vertices and edges, with weights being

the self-intersection numbers. So the matrix Q will be related to a Gram matrix of the minus-intersection form. Note that our choice of a matrix differs from the usual matrix of a weighted graph. This is done because the minus-intersection form is “almost” positive-definite.

Suppose Γ is a weighted forest. Denote by $E(\Gamma)$ and $V(\Gamma)$ the sets of edges and vertices of Γ respectively. Denote by $d(\Gamma)$ the determinant of $Q(\Gamma)$ (note that it does not depend on the ordering of the vertices). For every subset S of $E(\Gamma)$ denote by $Supp(S)$ the set of all vertices that are endpoints of some edge from S . A subset of S will be called disjoint if no two edges share a vertex (i.e. $|Supp(S)| = 2|S|$). Denote by $A(\Gamma)$ the set of all disjoint subsets of $E(\Gamma)$. The following lemma provides a useful formula for the determinant $d(\Gamma)$.

Lemma 5.1. *In the above notations, if the weight of $v \in V(\Gamma)$ is a_v ,*

$$d(\Gamma) = \sum_{S \in A(\Gamma)} \left((-1)^{|S|} \cdot \prod_{v \in V(\Gamma) \setminus Supp(S)} (-a_v) \right)$$

Proof. Using ordering of $V(\Gamma)$, we identify it with $\{1, 2, \dots, |V(\Gamma)|\}$. The determinant $d(\Gamma)$ is the sum of $|V(\Gamma)|!$ terms corresponding to permutations of $V(\Gamma)$. Note that such term is zero unless the corresponding permutation sends each i to itself or to a vertex j which is connected to i by an edge. Because Γ is a forest, the cycle decomposition of such permutation consists of fixed points and transpositions (i, j) , where (i, j) are edges of the graph. Obviously, these edges must be disjoint, so the set of such permutations is in one-to one correspondence with $A(\Gamma)$. The formula follows, considering that the sign of such permutation is determined by the parity of the number of transpositions. \square

We will use the following notations and definitions.

Definition 5.1. 1) For $P \in V(\Gamma)$ we denote by Γ_P the weighted forest obtained from Γ by removing P and all edges involving P . We define the determinant label d_P of the vertex P as $d(\Gamma_P)$.

2) For $(P, Q) \in E(\Gamma)$ we denote by Γ_{PQ} the weighted forest obtained from Γ by removing the edge (P, Q) . We define the determinant label d_{PQ} of the edge (P, Q) as $d(\Gamma_{PQ})$.

3) For a subset $\{P, \dots, Q\}$ of $V(\Gamma)$ we denote by $\Gamma_{\{P, \dots, Q\}}$ the weighted forest obtained from Γ by removing all vertices in this subset and all edges involving them. We denote by $d_{P, \dots, Q}$ the determinant of $\Gamma_{\{P, \dots, Q\}}$. **Note that for an edge (P, Q) the number $d_{P, Q}$ should not be confused with d_{PQ} .**

The following four properties of the determinants will be used extensively later. We will call them the multiplicativity property, the

expansion by vertex formula, the expansion by edge formula, and the weight increment formula.

Lemma 5.2. *Suppose Γ is a disjoint union of Γ_1 and Γ_2 . Then*

$$d(\Gamma) = d(\Gamma_1) \cdot d(\Gamma_2)$$

Proof. With the appropriate ordering of the vertices, the matrix $Q(\Gamma)$ is block-diagonal, with blocks being $Q(\Gamma_1)$ and $Q(\Gamma_2)$. \square

Lemma 5.3. *Suppose $P \in V(\Gamma)$. Then*

$$d(\Gamma) = -a_P \cdot d_P - \sum_{(P,Q) \in E(\Gamma)} d_{P,Q}$$

Proof. This follows from Lemma 4.1 by breaking up the terms in the sum for $d(\Gamma)$ into classes based on the way the vertex P is involved. \square

Lemma 5.4. *Suppose (P, Q) is an edge of Γ . Then*

$$d(\Gamma) = d_{PQ} - d_{P,Q}$$

Proof. This follows from Lemma 4.1 by breaking up the terms in the sum for $d(\Gamma)$ into two classes: $(P, Q) \notin S$ and $(P, Q) \in S$. \square

Lemma 5.5. *Suppose Γ' is obtained from Γ by decreasing the weight of P by 1. Then*

$$d(\Gamma') = d(\Gamma) + d_P$$

Proof. Expanding and contracting by the vertex P ,

$$\begin{aligned} d(\Gamma') &= (-a_P + 1) \cdot d_P - \sum_{(P,Q) \in E(\Gamma)} d_{P,Q} = \\ &= \left(-a_P \cdot d_P - \sum_{(P,Q) \in E(\Gamma)} d_{P,Q} \right) + d_P = d(\Gamma) + d_P \end{aligned}$$

\square

Generalizing the behavior of the weighted graphs of exceptional curves under blow-ups, we define the blow-up operations on the set of all weighted forests as follows. The blow-down is defined as the operation opposite to blow-up.

Definition 5.2. 1) *Suppose P is a vertex of Γ . Then the blow-up of Γ at P is a forest with the set of vertices $V(\Gamma) \cup \{R\}$, the set of edges $E(\Gamma) \cup \{(PR)\}$; $a_R = -1$, a_P is decreased by 1, all other weights do not change.*

2) Suppose (P, Q) is an edge of Γ . Then the blow-up of Γ at P is a forest with the set of vertices $V(\Gamma) \cup \{R\}$, the set of edges $E(\Gamma) \cup \{(P, R), (R, Q)\} \setminus \{(P, Q)\}$; $a_R = -1$, a_P and a_Q are decreased by 1, all other weights do not change.

Lemma 5.6. *The determinant of the graph is preserved by blow-ups and blow-downs.*

Proof. Obviously, it is enough to prove this for the blow-ups.

Case 1. Suppose $P \in V(\Gamma)$ is blown up, and the new vertex is R . Suppose the new graph is Γ' . Expanding by vertex R , we get

$$d(\Gamma') = d(\Gamma'_R) - d(\Gamma'_{PR}) = d(\Gamma'_R) - d(\Gamma_P)$$

By the weight increment formula,

$$d(\Gamma'_R) = d(\Gamma) + d(\Gamma_P),$$

which implies the result.

Case 2. Suppose $(P, Q) \in E(\Gamma)$ is blown up, and the new vertex is R . Suppose the new graph is Γ' . Because Γ is a forest, Γ_{PQ} is a disjoint union of two forests. We will denote the one that contains P by $P\Gamma$ and the one that contains Q by $Q\Gamma$. Similarly, Γ'_R is a disjoint union of $P\Gamma'$ and $Q\Gamma'$.

Expanding by the vertex R , we get

$$d(\Gamma') = d(\Gamma'_R) - d(\Gamma'_{P,R}) - d(\Gamma'_{Q,R})$$

By the multiplicativity property,

$$d(\Gamma'_R) = d(P\Gamma') \cdot d(Q\Gamma'),$$

$$d(\Gamma'_{P,R}) = d(P\Gamma'_P) \cdot d(Q\Gamma'),$$

$$d(\Gamma'_{Q,R}) = d(Q\Gamma'_Q) \cdot d(P\Gamma').$$

Note that $P\Gamma'_P = P\Gamma_P$. By the weight increment formula,

$$d(P\Gamma') = d(P\Gamma) + d(P\Gamma_P).$$

Likewise, $d(Q\Gamma') = d(Q\Gamma) + d(Q\Gamma_Q)$. Putting this together, we get

$$\begin{aligned} d(\Gamma') &= (d(P\Gamma) + d(P\Gamma_P))(d(Q\Gamma) + d(Q\Gamma_Q)) - d(P\Gamma_P)(d(Q\Gamma) + d(Q\Gamma_Q)) - \\ &\quad - d(Q\Gamma_Q)(d(P\Gamma) + d(P\Gamma_P)) = d(P\Gamma)d(Q\Gamma) - d(P\Gamma_P)d(Q\Gamma_Q) \end{aligned}$$

On the other hand, expanding $d(\Gamma)$ by the edge (P, Q) , we get

$$d(\Gamma) = d(\Gamma_{PQ}) - d(\Gamma_{P,Q}).$$

By the multiplicativity property, this equals the above expression for $d(\Gamma')$, which completes the proof of the lemma. \square

Corollary 5.1. *The determinants of vertices and edges are preserved by blow-ups and blow-downs that do not destroy them.*

Proof. Most blow-ups and blow-downs, that do not destroy the corresponding vertex or edge, do not involve it. For those that do, we note that blowing up an edge (P, Q) and then removing P is the same as removing P and then blowing up Q . Also, when a vertex P is blown-up and then removed, the determinant of the resulting graph is the same as the determinant of the original graph with vertex P removed, because the new vertex has weight -1 . \square

The above corollary means that just like the coefficients of the \bar{K} labels, the determinant labels on vertices and edges of the graph of exceptional curves on Z do not change once the point or edge is created (the edge, however, may get destroyed). While we are ultimately more interested in the labels of vertices, we need to keep track of the edges as well. The following lemma is very significant, as it describes the determinants of vertices and edges formed after the blow-up in terms of the nearby determinants and the determinant of the entire graph. The existence of these formulas is not intuitively obvious, and the formulas themselves are very important for applications.

Lemma 5.7. *Suppose Γ' is obtained from Γ by a single blow-up. Suppose $d(\Gamma') = d(\Gamma) = d$.*

Case 1. Suppose $P \in V(\Gamma)$ is blown up, and the new vertex is R . Then $d_R = d_P + d$ and $d_{PR} = d_P + d$. Here d_R and d_{PR} are the determinants of a vertex and an edge in Γ' , and d_P is the determinant in Γ or Γ' (which are the same by the previous lemma).

Case 2. Suppose $(P, Q) \in E(\Gamma)$ is blown up, and the new vertex is R . Then

$$d_R = 2d_{PQ} + d_P + d_Q - d,$$

$$d_{PR} = d_P + d_{PQ},$$

$$d_{QR} = d_Q + d_{PQ}.$$

Proof. In Case 1, one immediately sees that $d_{PR} = d_R$, because the weight of R is (-1) . Then the weight increment formula implies that $d_R = d + d_P$.

In Case 2, using the notation and work from the previous lemma, we have

$$\begin{aligned} d_R &= d(\Gamma'_R) = d(P\Gamma')d(Q\Gamma') = (d(P\Gamma) + d(P\Gamma_P)) \cdot (d(Q\Gamma) + d(Q\Gamma_Q)) = \\ &= d_{PQ} + d_P + d_Q + d(P\Gamma_P)d(Q\Gamma_Q) = d_{PQ} + d_P + d_Q + (d_{PQ} - d). \end{aligned}$$

This proves the formula for the vertex. For the edges, it is enough to prove the formula for d_{PR} .

$$d_{PR} = d(P\Gamma') \cdot d(Q\Gamma) = (d(P\Gamma) + d(P\Gamma_P))d(Q\Gamma) = d_{PQ} + d_P. \quad \square$$

In what follows, the weighted forest Γ will be the tree of exceptional curves on Z , which is a resolution of singularities of a counterexample to the Jacobian Conjecture with $X = P^2$. We define the determinant label of an exceptional curve P as d_P . To help distinguish the different labeling, we will call the coefficient of P in \bar{K}_Z the \bar{K} -label of P .

The relevance of the determinant labeling stems from the following theorem.

Theorem 5.1. *Suppose T is any collection of type 1 curves, and Γ_T is the subgraph obtained from Γ by removing the curves from T and all affected edges. Then $d(\Gamma_T) < 0$. In particular, $d_P < 0$ for any type 1 curve P .*

Proof. Because of the Hodge Index Theorem, if $d(\Gamma_T) \geq 0$, the restriction of the intersection form to the span of the classes of exceptional curves not in T is negative semi-definite. On the other hand, the Ample Ramification Theorem implies that some linear combination of curves of type 2, 3, 4 on Z has positive self-intersection. \square

Theorem 5.2. *Every \bar{K} -negative curve P with negative d_P has a \bar{K} -0 curve as its ancestor.*

Proof. For a curve P , we will denote its \bar{K} label by b_P . We will identify the curves with vertices on the graphs. Suppose we perform any number of blow-ups without creating a $\bar{K} - 0$ curve. This means that $b_P \leq -1$ for all curves P involved. We will prove by induction on the number of blow-ups the following two conditions.

1) For all vertices P , $d_P + b_P \geq -1$. Note that this implies that $d_P \geq 0$.

2) For all edges PQ , $d_{PQ} \geq 0$.

The base of induction is easy to check. To prove the step, we need to consider two cases of blow-ups. Note that $d(\Gamma) = d = -1$.

Case 1. Blowing up a vertex P to get a new vertex R .

$$d_R + b_R = (d_P - 1) + (b_P + 1) = d_P + b_P \geq -1; \quad d_{PR} = d_R \geq 0.$$

Case 2. Blowing up an edge PQ to get a new vertex R .

$$\begin{aligned} d_R + b_R &= 2d_{PQ} + d_P + d_Q + 1 + b_P + b_Q = \\ &= (d_P + b_P) + (d_Q + b_Q) + 1 + 2d_{PQ} \geq (-1) + (-1) + 1 + 0 = -1. \end{aligned}$$

For the edges,

$$d_{PR} = d_P + d_{PQ} \geq 0 + 0 = 0.$$

□

Example 5.1. *Starting from $X = P^2$, we blow up a point at infinity, and then a point on the new curve. The resulting surface has the following graph of exceptional curves.*

$$\begin{array}{ccccc} 1 & & 0 & & -1 & -1 \\ \circ & \text{---} & \circ & \text{---} & \circ & \\ -2 & & -1 & & 0 & \\ 0 & & -2 & & -1 & \end{array}$$

Here the labels above the graph are the edge and vertex determinants, the labels right below the graph are the \bar{K} -labels, and the labels on the bottom line are the self-intersection labels. If one blows up the point of intersection of the last two blown-up curves, one gets a curve with negative \bar{K} -label and negative determinant label. Other such curves could be obtained by further blowups.

The following theorem shows that, up to a polynomial automorphism of the original $A^2 \subset X$, all type 1 curves are obtained as in the example above.

Theorem 5.3. *Suppose Z is a resolution of singularities of a counterexample to the Jacobian Conjecture. Suppose P is \bar{K} -negative curve with negative d_P . Suppose E is the first $\bar{K} - 0$ curve that has to be created to create P (the oldest $\bar{K} - 0$ ancestor of P). Then there exists a sequence of blow-ups and blow-downs at infinity, that transform Z into Z' , which has the same graph as the example above, with E being the $\bar{K} - 0$ curve. (Note that the curve with the \bar{K} -label (-2) may differ from the original pullback of infinity on Z).*

Proof. We will follow the way P was created, changing the pullback of infinity as needed, to decrease the number of ancestors of E . The key idea is that one can rule out the possibility of creating P in some choices of blow-ups.

We start with blowing up a point at infinity. If then the new point is blown-up on the last curve, we are in the situation of the example above, and the result is obviously true. So the only non-trivial option left is to blow the point of intersection of the two curves, creating the following graph:

$$\begin{array}{ccccc} 1 & & 2 & & 0 \\ \circ & \text{---} & \circ & \text{---} & \circ \\ -2 & & -3 & & -1 \\ -1 & & -1 & & -2 \end{array}$$

The next step in the sequence of creation of E must either blowing up the left edge, the middle point, or the right edge of the above graph.

If the left edge is blown up, we end up in the part of the graph with all positive edge labels, and the inequality $d_Q + b_Q \geq 0$ satisfied for all the vertices (except the one that used to be the middle vertex). The induction argument, similar to the one in the proof of the last theorem, implies that even if we create a $\bar{K} - 0$ curve, its determinant label, and the label of the edge created with it, will be non-negative. Since we have to go back to the “negative \bar{K} ” territory, we will never get a curve with negative \bar{K} and determinant labels.

If the middle point is blown (which corresponds to blowing up a new point on the last blown-up curve) then the newly blown curve can serve as a new pull-back of infinity, which is closer to E .

So the only case left is when the right edge is blown up. After that we again have three possibilities. Blowing up the left edge can be discarded the same way as above. If we blow up the last vertex, we will be forced to blow up the newly blown-up vertex, creating a new curve that can serve as a pull-back of infinity.

In general, suppose the creation sequence for E starts with blowing up the right edge exactly k times. This leads to the following graph:

$$\begin{array}{ccccccc} \overset{1}{\circ} & \overset{1}{\text{---}} & \overset{2}{\circ} & \overset{2}{\text{---}} & \dots & \overset{(k+2)}{\circ} & \overset{0}{\text{---}} & \overset{0}{\circ} \\ -2 & & -3 & & & -(k+3) & -1 & \\ -1 & & -2 & & & -1 & -(k+2) & \end{array}$$

After this we are forced to blow up the second from the right vertex, and then blow up the resulting vertex, and so on, until we get a curve with the \bar{K} -label (-2) . Notice that this curve Q can serve as a new pull-back of infinity, because all other curves can be contracted one after another to a smooth surface, which has to be P^2 , since the anticanonical class is $-3Q$ and $Q^2 = 1$. Note that this Q is closer to E than the original pull-back of infinity. \square

We are now going to extend the above results to several \bar{K} -negative curves, with the condition that removing them all from the graph of exceptional curves produces a graph with negative determinant.

First, we need to introduce some new notation. Suppose $X = P^2$ is the standard compactification of A^2 , Z is obtained from X by a sequence of blowups outside A^2 ; Γ is the (dual) graph of the exceptional curves on Z , $E = \pi^{-1}(\infty)$, where $\pi : Z \rightarrow X$ is the natural map. Suppose P is an exceptional curve on Z , and a vertex of Γ . We denote by d'_P the determinant of $\Gamma_{E,P}$. We denote by u_P the coefficient of P in the formula that expresses $\pi^*(\infty)$ in terms of the exceptional curves on Z : $\pi^*(\infty) = E + \sum_{P \neq E} u_P P$. Note that u_P is a natural number.

Lemma 5.8. *In the above notation, $u_P^2 = d_P + d'_P$.*

Proof. While an inductive argument is possible, the following direct proof is easier. Define $L = E + \dots + U_P P + \dots$, the class of the pullback of (∞) on Z . Note that L intersects by 1 with E and by 0 with all other exceptional curves. If we change a basis of the lattice of the divisor classes on Z , replacing P by L , the Gram matrix of the minus-intersection form will be multiplied by u_P^2 . Thus

$$(-1) \cdot u_P^2 = \det \begin{pmatrix} (-E^2) & \dots & -1 \\ \dots & & \\ -1 & & -1 \end{pmatrix},$$

where the last row and column of the matrix on the right correspond to L and the first correspond to E . Note that all the coefficients in the last row and column are zero unless indicated, and that removing these row and column produces the matrix of Γ_P . Using cofactor expansion w.r.t. the last row, and then (for one of the terms) the first row, we get $-u_P^2 = -d_P - d'_P$. \square

Lemma 5.9. *Suppose P and Q are on the different sides of E (i.e. E belongs to the pass in the tree Γ that connects P and Q). Then the determinant $d_{P,Q}$ of the graph $\Gamma_{P,Q}$ is given by the following formula:*

$$d_{P,Q} = u_P u_Q - d_P d_Q$$

Proof. We will denote $d_{P,Q}$ by ${}_P d_Q$. The notation ${}_{C_1 C_2} d_Q$ will stand for the determinant of the graph obtained by removing the vertex Q and the edge $C_1 C_2$.

We first perform a sequence of blowups to create Q , without producing any ancestors of P . Then we perform a sequence of blowups that creates P , keeping track of the determinants of vertices and edges of the new graph with the point Q removed. Note the following:

- 1) ${}_E d_Q = d'_Q$;
- 2) $d(\Gamma_Q) = d_Q$;
- 3) For all ancestors C of P and all edges $C_1 C_2$ involved in creating P , their determinant labels ${}_C d_Q$ and ${}_{C_1 C_2} d_Q$ are linear combinations of d'_Q and d_Q . Let us denote the coefficients of these linear combinations by a_C , b_C , $a_{C_1 C_2}$, $b_{C_1 C_2}$ so that

$${}_C d_Q = a_C \cdot d'_Q + b_C \cdot d_Q, \quad {}_{C_1 C_2} d_Q = a_{C_1 C_2} \cdot d'_Q + b_{C_1 C_2} \cdot d_Q$$

- 4) From the formulas for the determinants of blowups (Lemma 5.7) the coefficients a_C and $a_{C_1 C_2}$ behave like d_C and $d_{C_1 C_2}$ for a graph with the determinant $d = 0$, and initial value 1. By induction, one can easily prove that $a_C = d_C + d'_C$, which equals u_C^2 .

5) From the formulas for the determinants of blowups (Lemma 5.7) the coefficients b_C and $b_{C_1C_2}$ behave like d_C and $d_{C_1C_2}$ for a graph with the determinant $d = 1$, and initial value 0. By induction, one can easily prove that $b_C = d'_C$.

6) So, ${}_Cd_Q = u_C^2 \cdot d'_Q + d'_Q = u_C^2(u_Q^2 - d_Q) + (u_C^2 - d_C)d_Q = u_C^2u_Q^2 - d_Cd_Q$
 \square

Lemma 5.10. *In the above notation, for a curve P with \bar{K} label $\bar{k}_P < 0$ and $d_P < 0$ the following inequality is satisfied:*

$$u_P^2 \geq |d_P|$$

Proof. Because $d_P < 0$, we need to prove that $u_P^2 + d_P \geq 0$. Note that $u_P^2 + d_P = 2d_P + d'_P$. For all ancestors C of P , define $l_C = 2d_C + d'_C$ and $l_{C_1C_2} = 2d_{C_1C_2} + d'_{C_1C_2}$. The labels l_C and $l_{C_1C_2}$ behaves like those in the graph with $d = -1$. Like in the proof of Theorem 5.2., one can prove by induction the following two statements:

$$1) l_C + \bar{k}_C \geq -1$$

$$2) l_{C_1C_2} \geq 0$$

So $l_P + \bar{k}_P \geq -1$. Because $\bar{k}_P < 0$, $l_P \geq 0$. \square

Theorem 5.4. *Suppose P and Q have negative \bar{k} labels and determinant labels and lie on the different sides of E . Then $d_{P,Q} \geq 0$.*

Proof. This follows directly from Lemma 5.9 and Lemma 5.10. \square

Corollary 5.2. *Suppose $\{E_1, \dots, E_k\}$ is any collection of exceptional curves of Z with negative \bar{k} labels, such that removing them all from the graph of exceptional curves produces a graph with negative determinant. Then they all lie on the one side of E .*

Remark 5.1. *The above Corollary can be applied to the set of all curves of type 1, as defined in sections 1–3. As a result, we conclude that all of these curves are on one side of $\pi^{-1}(\infty)$.*

6. CHANGING THE TARGET SURFACE

In order to better understand the structure of a possible counterexample to the Jacobian Conjecture, one should not be content with just having the P^2 as the compactification of the target plane. Indeed, we proved in Section 3 that for the compactification $Y = P^2$ all curves that are contracted to a point at infinity (including the pull-back of the line at infinity) are mapped to one point. So all the “action” is in (or, rather, above) this point, and we should blow it up to better understand what is going on. In fact, it makes sense to keep blowing up the image of the pull-back of the line at infinity until this image becomes a curve.

In what follows, we will use the following conventions: X is compactification of the origin A^2 ($X = P^2$, unless specified otherwise), Y is the compactification of the target A^2 , Z is the resolution of the corresponding map, $Z \rightarrow W \rightarrow Y$ is the Stein factorization. The maps are $\pi : Z \rightarrow X$, $\varphi : Z \rightarrow Y$, $\tau : Z \rightarrow W$, and $\rho : W \rightarrow Y$. When we consider several compactifications at the same time, these varieties and maps acquire matching indices (e.g. ρ_3 is the map from W_3 to Y_3). The log-ramification divisor on W is $\bar{R} = \sum r_i R_i$, where R_i are all di-critical curves and r_i are the corresponding ramification indices.

Definition 6.1. For $n \geq 0$ we denote by \mathcal{F}_n the smooth compactification of A^2 that contains exactly two exceptional curves, F_0 and F_n , that are smooth and satisfy $F_0 F_n = 1$, $F_n^2 = 0$, $F_0^2 = -n$.

These compactifications \mathcal{F}_n appear naturally as the compactifications Y of the target A^2 . For example, starting from $Y = P^2$ and blowing up $\varphi(\pi^{-1}(\infty))$, one gets \mathcal{F}_1 . The curve F_1 is the strict pull-back of the line at infinity on Y and the curve F_0 is the blown-up curve. One can show (see below) that on this new compactification, that we will call Y_1 , the map φ_1 sends $\pi_1^{-1}(\infty)$ to the point of intersection of F_0 and F_1 . (Here π_1 is the map from Z_1 to $X_1 = X = P^2$). Easy calculations show that the \bar{K} -labels of F_0 and F_n are -1 and $-(n+1)$ respectively.

Theorem 6.1. Suppose we have any compactification of a counterexample to the Jacobian Conjecture with $X = P^2$ and $Y = \mathcal{F}_n$. Then $\varphi(\pi^{-1}(\infty))$ must be a point on F_n (it may be the point of intersection of F_0 and F_n). Moreover, all di-critical curves on W pass through $\tau(\pi^{-1}(\infty))$.

Proof. If $\varphi(\pi^{-1}(\infty))$ is F_0 or a point on F_0 , then $\varphi^*(F_n)$ is a linear combination of exceptional curves on Z that does not include $\pi^{-1}(\infty)$. This is impossible because $(\varphi^*(F_n))^2 = (\deg(\rho)) \cdot F_n^2 = 0$.

Recall that we have at least one di-critical curve. As in Theorem 3.2, consider the intersection of a di-critical curve R_k on W with the log-canonical class. We use the adjunction formula for ρ and the adjunction inequality for R_k on W from Section 4. Here $\bar{R} = \sum_i r_i R_i$ is the di-critical ramification divisor on W , $\rho(R_k)$ is the reduced image of R_k .

$$-1 \leq R_k \cdot \bar{K}_W = f_k \rho(R_k) \cdot \bar{K}_Y + R_k \cdot \bar{R}$$

Note that $\rho(R_k)$ must intersect at least one of the curves F_0 and F_n , so $\rho(R_k) \cdot \bar{K}_Y \leq -1$. So $R_k \cdot \bar{R} \geq 0$. Thus for every point w on W that is an intersection of a di-critical curve with the other exceptional curves,

$(\sum_{i|w \in R_i} r_i R_i)^2 \geq 0$, which implies that $\pi^{-1}(\infty)$ is among the irreducible components of $\tau^*(\sum_{i|w \in R_i} r_i R_i)$, thus $\tau(\pi^{-1}(\infty)) = w$. \square

Suppose we have a counterexample to the Jacobian conjecture, compactified and resolved, so that X and Y are isomorphic to P^2 . Blowing up the point $\varphi(\pi^{-1}(\infty))$, we get a compactification $Y_1 = \mathcal{F}_1$. If the point $\varphi_1(\pi_1^{-1}(\infty))$ is the intersection of the exceptional curves F_0 and F_1 on Y_1 , we blow it up and contract the strict preimage of F_1 , obtaining the compactification $Y_2 = \mathcal{F}_2$. If the point $\varphi_2(\pi_2^{-1}(\infty))$ is the intersection of the exceptional curves F_0 and F_2 on Y_2 , we again blow it up and blow down the strict pullback of the exceptional fiber to get $Y_3 = \mathcal{F}_3$, and so on. We will continue to do this until for some n we get $\varphi_n(\pi_n^{-1}(\infty))$ to be on F_n and not on F_0 .

Then we blow up $\varphi_n(\pi_n^{-1}(\infty))$ on Y_n . Note that $n \geq 2$, so the \bar{K} -label of the newly blown-up curve is less than or equal to -2 . Thus the same argument as in the theorem above implies that the pullback of the line at infinity is mapped by τ to a point, and all di-critical curves pass through it. The image by ρ of this point must be on the newly blown-up curve. If it is not on the strict pullback of F_n , we can blow down the strict pullback of F_n , so that the new compactification is \mathcal{F}_{n-1} . We will keep doing this until $\varphi(\pi^{-1}(\infty))$ is on the point of intersection of pullback of the exceptional fiber and the newly blown-up curve. This implies that before the last blowup the images of all di-critical components on Y “touched” the exceptional fiber. (The quotation marks are due to the fact that these images are uni-branched, but not necessarily smooth at $\varphi(\pi^{-1}(\infty))$).

The following theorem determines the structure of the variety W_n and the map ρ_n in the case when $\varphi_n(\pi_n^{-1}(\infty))$ is a point on F_n and not on F_0 .

Theorem 6.2. *Suppose $X = P^2$, $Y = \mathcal{F}_n$, and $\varphi_n(\pi_n^{-1}(\infty))$ is a point on $F_n \setminus F_0$. We denote by R_k the di-critical curves on W , by E_j the preimages of F_n and by D_i the preimages of F_0 . Then the following are true.*

- 1) *All E_j contain $w = \rho(\pi^{-1}(\infty))$.*
- 2) *Every D_i intersects exactly one E_j at a cyclic quotient singularity and is smooth elsewhere (generalized simple normal crossing). The map ρ is degree one on each D_i (but it may be ramified along D_i).*
- 3) *All R_k contain w .*
- 4) *There are no other points of intersections of exceptional curves on W .*

Proof. Statement (3) was proven in Theorem 6.1. Statement (2) follows from Theorem 4.6 and Theorem 4.7 (a). To prove statement (1) note that if some E_j does not contain w , it cannot intersect with any R_k . So Theorems 4.6 and 4.7(a) apply to it. Because $E_j \rho^*(F_0) > 0$, it must intersect with some D_i . Since that D_i can intersect no other exceptional curve, and neither can E_j , we get a contradiction with the connectedness of the set of exceptional curves on W . Finally, statement (4) follows from the fact the set of exceptional curves on W is an image of a tree of curves on Z . \square

Remark 6.1. 1) Note that in the above theorem we have not used the fact that the images of the di-critical curves “touch” F_n . Perhaps this can be explored by studying W for the surface Y obtained by blowing up $\rho(w)$.

2) One can calculate $E_j \cdot F_i$ for intersecting E_i and F_j based on the ramification indices.

3) If $f_j = 1$, then E_j must be smooth outside w (obtained by applying Theorem 4.6 after blowing up $\varphi(\pi^{-1}(\infty))$ several times); in this case there must be only one F_i intersecting this E_j . For $f_j > 1$ there may be several F_i intersecting E_j and E_j may have singularities at the points that are mapped by ρ to $\rho(w)$.

We will now continue to blow up $\varphi(\pi^{-1}(\infty))$ until it becomes a curve F on Y . Note that because the \bar{K} -label of $E = \pi^{-1}(\infty)$ is -2 , the \bar{K} -label of F can only be -1 or -2 , depending on whether the ramification index at E is 2 or 1. Thus, no curves with positive \bar{K} -label are ancestors of F . So Y obtained by this procedure only contains curves with non-positive \bar{K} -labels. The following theorem indicates that it must contain $\bar{K} = 0$ curves. Please note a notation change: the curves E_i and F_i now mean non-di-critical exceptional curves on W and Y respectively.

Theorem 6.3. Suppose Y , E and F are the varieties described above, and $X = P^2$, Z , W are the corresponding varieties, with the usual notation for the morphisms between them. Then for every di-critical curve R_k on W the following are true.

- 1) The point of intersection of $\rho(R_k)$ with $Y \setminus A^2$ belongs to exactly one exceptional curve F_k .
- 2) This curve F_k has \bar{K} -label 0.
- 3) The curve F_k intersects exactly one other exceptional curve, with \bar{K} -label -1 .

Proof. Suppose $\rho(R_k)$ intersects $Y \setminus A^2$ at a point y_k , and R_k intersect the union of other exceptional curves on W at w_k . Then if y_k

belong to at least one curve with negative \bar{K} -label,

$$-1 \leq R_k \cdot \bar{K}_W = R_k \cdot \rho^*(\bar{K}_Y) + R_k \cdot \bar{R} \leq -1 + R_k \cdot \bar{R}$$

Therefore $R_k \cdot \bar{R} \geq 0$. The same is true for all di-critical curve passing through w_k , so

$$\left(\sum_{j|w_k \in R_j} r_j R_j \right)^2 \geq 0$$

This implies that $\left(\sum_{j|w_k \in R_j} \tau^*(r_j R_j) \right)^2 \geq 0$, which contradicts the fact that the determinant label of E is positive. The same argument applies at any intermediate step to the curves R_k that do not contain $\tau(\pi^{-1}(\infty))$.

When a $\bar{K} - 0$ curve is created on Y , it must be a result of blowing up a point on a curve with \bar{K} -label -1 . At this step, $\varphi(\pi^{-1}(\infty))$ has to be the point of intersection of these two curves, otherwise we will get into the positive \bar{K} territory on Y . So the statement of the theorem is true for R_k at the intermediate step when $\rho(R_k)$ no longer contains $\varphi(\pi^{-1}(\infty))$; it is easy to see that the subsequent blowups of $\varphi(\pi^{-1}(\infty))$ do not change that. \square

Definition 6.2. *For a hypothetical counterexample to the Jacobian Conjecture, we call the varieties X, Y, Z, W and morphisms between them, obtained by the sequence of blowups on Y of $\varphi(\pi^{-1}(\infty))$ until it becomes a curve, its resolution of the image of the line at infinity.*

Remark 6.2. *The varieties in the resolution of the image of the line at infinity differ slightly from those described in Theorem 6.3, because we do not blow down any curves on Y , as we did during the modifications of \mathcal{F}_n . However, the proof and the conclusions of Theorem 6.3 remain true. Note also that the exceptional curves on Y can be enumerated w.r.t. to the order of their creation. In particular, we can talk about the last $\bar{K} - 0$ curve there. Note that the resulting Y and W are completely determined by the original pair of polynomials, while Z may vary.*

The above discussion allows to give an alternate proof of the Remark 5.1. Combined with the “One Point at Infinity” theorem of Abhyankar (cf. [1]) and the results of section 4, it gives the following result.

Theorem 6.4. *Suppose a counterexample to the Jacobian Conjecture exists. Then there exists another counterexample, such that its resolution of the image of the line at infinity has the following properties, in addition to those described in Theorem 6.3.*

1) *If F_l is the last $\bar{K} - 0$ curve on Y , then at least one image of a di-critical curve intersects it.*

2) The curve $F = \varphi(\pi^{-1}(\infty))$ on Y is a curve of valency two.

3) Removing F from the graph of exceptional curves on Y produces two subgraphs. One of them contains all the exceptional curves of type 1 in the sense of sections 1–3. The other contains the curve F_l and one of the following is true about it:

Case A) It consists of a single curve F_l ; then $\varphi(\pi^{-1}(\infty))$ is the $\bar{K} - (-1)$ curve it is connected to.

Case B) It consists of two curves: F_l and the $\bar{K} - (-1)$ curve it is connected to; then $\varphi(\pi^{-1}(\infty))$ is a $\bar{K} - (-2)$ curve, connected to the $\bar{K} - (-1)$ neighbor of F_l .

Proof. The “One Point at Infinity” theorem of Abhyankar (cf. [1]) implies that if for a given counterexample to the two-dimensional Jacobian Conjecture the strict preimage of a generic line L on the target P^2 is a curve on the origin P^2 with only one point at infinity, one can produce (precomposing with a suitable polynomial automorphism) a “smaller” counterexample. In our terminology, the points of infinity are the points $\pi(\rho^{-1}(A_i))$, where A_i are points of intersection of L and the images of the di-critical curves, and the intersection of L with the line at infinity on the target P^2 . Thus for a given counterexample, we can precompose it with the polynomial automorphism to get another counterexample, where not all $\pi(\rho^{-1}(R_i))$ and $\pi(\rho^{-1}(\infty))$ are the same point. This is equivalent to the condition that not all di-critical curves and curves of type 1 in the terminology of sections 1–3 are on the same side of the curve $\pi^{-1}(\infty)$ on X . Note that this condition is preserved by the subsequent modifications of X . Note that by the Remark 5.1 all curves of type 1 must lie on one side of the curve $\pi^{-1}(\infty)$ on X . So at least one di-critical curve does not lie in the same connected component of the tree obtained by removing $\pi^{-1}(\infty)$.

Suppose F' is the last created $\bar{K} - 0$ curve on Y that intersects with the images of the di-critical curves. We claim that all points blown up after its creation were points of intersection of two exceptional curves on Y . Indeed, otherwise all images of the di-critical curves on Y and the strict pull-back of the line at infinity of the original target plane would have been in the same connected component of the tree of the exceptional curves with $\varphi(\pi^{-1}(\infty))$ removed. Because we stop the process as soon as $\varphi(\pi^{-1}(\infty))$ is a curve, its valency is 1 or 2. If its valency is 1, then Theorem 4.7 implies that $E = \tau(\pi^{-1}(\infty))$ is an end-curve in the graph of exceptional curves on W , which contradicts our assumption that not all di-critical curves and curves mapped to the original line at infinity are in the same connected component of the graph of exceptional curves on X . If the valency is 2, then Theorem

4.7 implies that the valency of E is two. Because all exceptional curves on Y are ancestors of F , the connected component of the graph of exceptional curves on Y that does not contain the di-critical curves and the target line at infinity is a chain. Therefore Theorem 4.7 implies that every exceptional curve on W that is mapped by ρ to any of these curves has the same valency (one or two) as its image. Therefore the map ρ sends the connected component of the tree of curves on W without E which does not contain the curves of type 1 to this chain of curves. This again implies that all di-critical components and curves of type 1 are on the same side of E , a contradiction.

Therefore, after F' was created we could only blow up points of intersections of two curves. This immediately implies that $F' = F_l$. Recall that the \bar{K} label of F is -1 if the ramification at E is 2 and -2 if the ramification at E is 1. One can easily check by looking at the \bar{K} labels on Y that it is possible that a point of intersection of F and its $\bar{K} - (-1)$ neighbor is blown up several times in the process, but no other points can be blown up until, possibly, one final blow-up that creates a $\bar{K} - (-2)$ curve. This implies the result. \square

7. INFORMAL DISCUSSION AND OBSERVATIONS

First of all, by further blowups on Y we can get a resolution on which the images of the di-critical curves intersect the other exceptional curves transversally. Such resolutions were considered by Domrına and Orevkov (cf. [5], [6]). Note that the maps of the splice diagrams that they considered have a nice geometric counterpart: if one contracts the chains of rational curves on Y to contract all curves of valency two, the curves on Y will have generalized simple normal crossings. Using the methods of section 4, one can easily show that the same will then be true for W , except for up to one cyclic quotient singularity on each of the di-critical curves. In a sense, our section 4 can be viewed as a strong argument for the importance of the splice diagrams. It puts a geometric structure on top of the combinatorial structure of Domrına and Orevkov. In particular, one can try to find a counterexample to the Jacobian Conjecture by finding a suitable map between splice diagrams and then constructing the corresponding finite map of singular compactifications of A^2 . The restrictions obtained in section 6 are strong, but not obviously prohibitive, and they may tell us where to look for possible counterexamples.

It may be interesting to investigate the connections between our approach and other results on the Jacobian Conjecture (see, e.g. [11], [13], [15]). It is also clear that we have barely tapped into the powerful

methods of modern birational geometry of surfaces. In particular, some ideas of the log Sarkisov Program (cf. [4]) may help to modify the possible counterexamples. It should also be noted that the theory of rational curves on rational surfaces, in particular the methods and results of the paper [10] may be helpful. Specifically, one can show that on Y there cannot exist a covering family of rational curves that intersect the union of the curves at infinity and the images of the dicritical curves at only one point. If $D = \sum F_i$ is the support of $\rho(\bar{R})$, and $\rho^*D = \bar{R} + G$ (i.e. G is the union of the co-exceptional curves on W) then on W we have $\bar{K}_W + G = \rho^*(\bar{K}_Y + D)$. Therefore some multiple of $\bar{K}_W + G$ must have a global section, an interesting condition to explore.

Finally, the following result seems to be true, but will appear elsewhere, once the details are clarified and confirmed.

Theorem 7.1. *Suppose a and b are any two fixed integers. Then there are only finitely many, up to the plane automorphisms, minimal graphs of exceptional curves that produce divisorial valuations with the center at infinity such that its determinant label is a and its \bar{K} label is b . In other words, such valuations are bounded, up to the plane automorphisms.*

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